then the conditions (2.8), determining the function λ in (2.7), can be replaced by the conditions

$$\lambda = \frac{1}{2} c\, \omega/\tau^2, \quad \omega = \left(1 - \frac{2}{3}\varepsilon\right)\sigma'_{ij} e'_{ij} - \tau\, d\,(\tau/\mu\alpha)/dt,$$
$$c = 0, \ \text{if} \quad f < 0 \ \text{or} \quad f = 0, \quad \omega \leqslant 0;$$
$$c = 1, \ \text{if} \quad f = 0, \quad \omega > 0.$$

For the computation of small increments of the stresses for a small interval of time from the deformation rates, instead of Eqs. (2.6)-(2.8), use can be made of a procedure, proposed in [4], for correction of the deviator of the stresses. Here the increments of the stresses before correction are calculated using Eqs. (1.11) and (1.16).

Using (1.12), (1.13), (1.15), and (2.6), for a medium with the condition (2.3), the equations of elastoplastic deformation can be formulated with a more general law of elastic deformation than in (2.6)-(2.8).

LITERATURE CITED

1. S. K. Godunov and E. I. Romenskii, "The unsteady-state equations of the nonlinear theory of elasticity in Euler coordinates," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1972).
2. S. S. Grigoryan, "The basic premises of soil mechanics," Prikl. Mat. Mekh., No. 6 (1960).
3. T. Y. Thomas, Plastic Flow and Fracture in Solids, Academic Press (1961).
4. M. L. Wilkins, "Calculation of elastoplastic flows," in: Computational Methods in Hydrodynamics [Russian translation], Mir, Moscow (1967).
5. L. I. Sedov, Introduction to the Mechanics of Continuous Media [in Russian], Fizmatgiz, Moscow (1962).
6. L. M. Kachanov, Principles of the Theory of Plasticity [in Russian], Nauka, Moscow (1969).

NONISOTHERMAL NONLINEAR WAVES IN A ROD MADE OF A DISSIPATIVE RUBBERLIKE MATERIAL

A. I. Leonov

UDC 532.5:532.135

Article [1] discussed in the isothermal approximation, a wave propagating in an elastoviscous rod and gave a numerical solution to the problem of the impact of a rod of finite length on a rigid barrier. With the presence of strong geometrical and physical nonlinearities in the determining equations, waves of very great intensity can be propagated in the rods, where the effects of nonisothermicity are considerable with the propagation of the waves. The present article is devoted to an investigation of these questions.

1. Basic Equations

With the study of the motion of the rods, as in [1], we shall use a description averaged over the cross section. The material of the rod is assumed to be incompressible with the density ρ_0.

The equations of the mass balance, momentum, and energy in a "rod approximation" have the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(fv) = 0, \quad \frac{\partial}{\partial t}(fv) + \frac{\partial}{\partial x}\left(fv^2 - \rho_0^{-1} f\sigma\right) = 0,$$
$$\frac{\partial}{\partial t}\{\rho_0 f (U + v^2/2)\} + \frac{\partial}{\partial x}\{\rho_0 f v (U + v^2/2\} = \frac{\partial}{\partial x}(fv\sigma - q) + \alpha \sqrt{f}(T - T_0), \tag{1.1}$$

where f is the area of the transverse cross section of the rod; v is the mean velocity over the cross section; σ is the mean normal stress over the cross section (determined as in the homogeneous case, using the condition of the reversion of the stresses to zero at the free surface of the rod); U is the specific internal energy; q is the longitudinal heat flux; T is the mean temperature over the cross section of the rod; T_0 is the tempera-

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 135-145, May-June, 1978. Original article submitted April 7, 1977.

0021-8944/78/1903-0390$07.50

ture of the surrounding medium; $\alpha > 0$ is the coefficient of lateral heat transfer; x is the longitudinal coordinate; and t is the time.

To close the system (1.1), equations describing the thermodynamic and rheological behavior of the medium must be formulated.

Let the rod consist of a dissipative rubberlike material of the type of raw (unlinked) rubber or a polymer melt. In this case, it can be postulated that its locally equilibrium thermodynamic state is similar to the state of linked rubber. We shall use the concept of an "ideal" incompressible rubber [2, 3], for which, in the approximation under consideration, the specific thermodynamic potentials (the internal energy U, the entropy S, and the free energy F) are determined by the expressions

$$U = U_0 + c_\lambda (T - T_0),\quad S = S_0 - (\mu'/\rho_0)\psi(\lambda) + c_\lambda \ln (T/T_0),$$
$$F = F_0 + (c_\lambda - S_0)(T - T_0) - c_\lambda T \ln (T/T_0) + (\mu' T/\rho_0)\psi(\lambda), \tag{1.2}$$

where λ is an "equilibrium" thermodynamic parameter, determining the elastic (highly elastic) deformation in the rod. In the case of homogeneous monaxial deformation, the quantity λ is the ratio of the length of the sample at a given moment of time to its length after instantaneous unloading. From this it follows that, for compression, $0 < \lambda < 1$; for elongation, $\lambda > 1$. In (1.2), μ' is a constant; here $\mu' T \sim E$, where E is the Young modulus; c_λ is the specific heat capacity for $\lambda = \text{const}$, which, over a rather broad interval of temperatures, can be assumed constant.

The dimensionless function $\psi(\lambda)$ in (1.2) is some elastic potential. In the case of the potential of the network theory of high elasticity [2]

$$\psi(\lambda) = \lambda^2 + 2\lambda^{-1} - 3,\quad \mu' = (1/2)\rho_0 N k, \tag{1.3}$$

where N is the number of effective molecular chains in the network per unit of volume; k is the Boltzmann constant.

The potential (1.3) describes rather well the mechanical properties of rubbers with not too great elongations $\lambda < 2$-3. For very large values of λ, an empirical potential [4], describing well the mechanical properties of rubbers with large deformations, must be used:

$$\psi(\lambda) = (2/n)(\lambda^n + 2\lambda^{-n/2} - 3) + (b/n^m)(\lambda^n + 2\lambda^{-n/2} - 3)^m, \tag{1.4}$$

where n, b, and m are positive empirical constants.

The qualitative dependences of the potential $\psi(\lambda)$ and its first three derivatives, in accordance with formulas (1.3) and (1.4) (dashed and solid lines, respectively), are shown in Fig. 1, from which it can be seen that, for $\lambda \le \lambda_*$, where λ_* is a point of inflection of $\psi'(\lambda)$ for the potential (1.4), the classical potential describes the results of experiment with qualitative correctness.

An investigation of the simplest equations of viscoelastic media in the presence of arbitrary finite elastic deformations was made by the methods of nonequilibrium thermodynamics in [5, 6]. In the case of monaxial

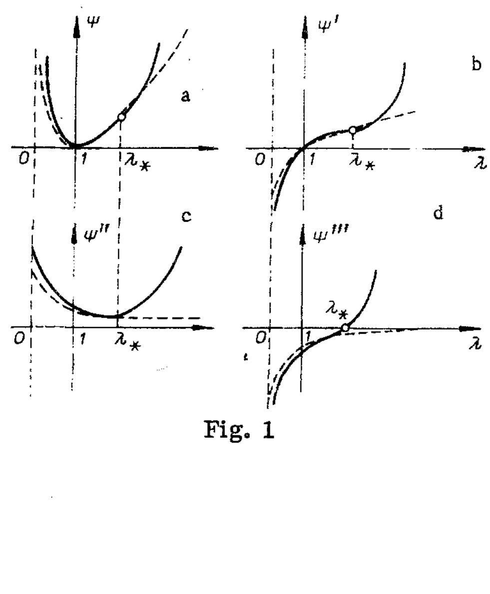

Fig. 1

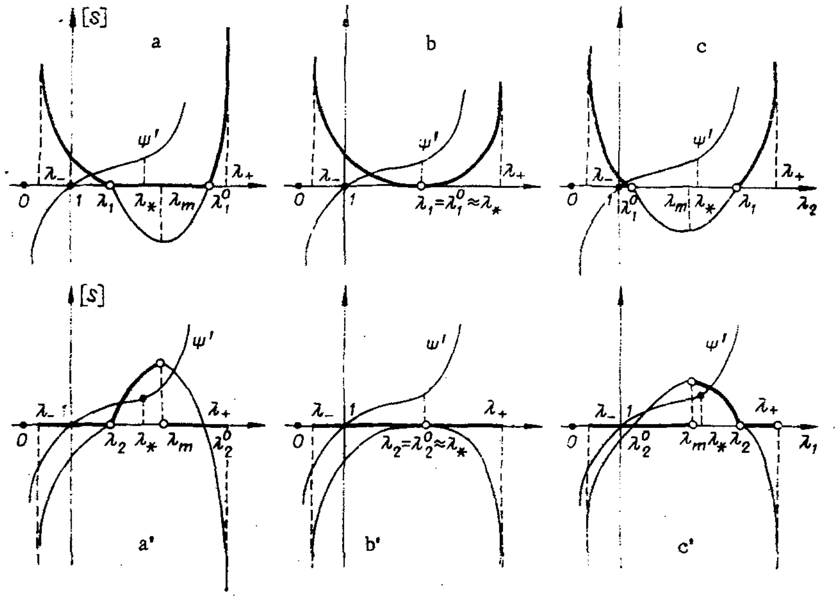

Fig. 2

elongation–compression, these equations, taking account of the specific properties of polymeric elastoviscous media, have the form

$$\sigma = \sigma_0(\lambda,\ T) + \xi(\lambda,\ T)\frac{\partial v}{\partial x},\ q = -\varkappa(\lambda,\ T)\frac{\partial T}{\partial x}\quad (\varkappa > 0); \tag{1.5}$$

$$\frac{1}{\lambda}\left(\frac{\partial \lambda}{\partial t} + v\frac{\partial \lambda}{\partial x}\right) + \frac{\varphi(\lambda)}{6\theta_0(T)} = \frac{\partial v}{\partial x}, \tag{1.6}$$

where $\sigma_0(\lambda,\ T)$ is the "equilibrium" stress; $\xi(\lambda,\ T)$ is the "Kelvin" viscosity, rising sharply with a rise in λ (or λ^{-1}) and falling sharply with a rise in T; $\varkappa$ is the coefficient of thermal conductivity. The second expression in (1.6) is the "kinematic resistance," where $\varphi(\lambda)/[6\theta_0(T)]$ is the rate of the irreversible deformations. Under these circumstances, as follows from [5, 6],

$$\begin{gathered}\sigma_0(\lambda,\ T) = \rho_0\lambda\left.\frac{\partial F}{\partial \lambda}\right|_T = \mu' T\lambda\psi'(\lambda),\\ \varphi(\lambda) = -\beta^{-1}\lambda\frac{d}{d\lambda}\exp\left\{-\frac{\beta}{2}[\psi(\lambda) + \psi(\lambda^{-1})]\right\},\\ \theta_0(T) = \theta'\exp(\Delta E/RT),\end{gathered} \tag{1.7}$$

where $\theta_0(T)$ is the characteristic relaxation time; ΔE is the activation energy; R is the gas constant; θ' is a constant; and the quantity $\beta(0 < \beta < 1)$ is a numerical parameter, taking account of the flexibility of the polymer chains.

For the investigation of waves of great intensity, in the determining equations (1.5) and (1.6) and in the energy equation [1.1] we shall omit the viscous term in the stress $\xi\partial v/\partial x$ and the term $q = -\varkappa\partial T/\partial x$, describing the longitudinal thermal conductivity, since outside of rather narrow zones where these quantities are small, and zones where these quantities are very great, they will be replaced, as in [1], by the surfaces of strong and weak discontinuities.

Thus, in what follows, in (1.1) and (1.5) we shall assume that

$$\sigma = \sigma_0(\lambda,\ T),\quad \xi(\lambda,\ T) \equiv 0,\quad q \equiv 0, \tag{1.8}$$

which corresponds to the nonlinear Maxwell rheological equations, used in the isothermal case in [1].

From (1.1) we can obtain an equation for the balance of the specific internal energy U, which, taking account of the expression for U from (1.2), has the form of the temperature balance:

$$\rho_0 c_\lambda\{\partial(fT)/\partial x + \partial(fvT)/\partial x\} = \sigma f\partial v/\partial x - \alpha\sqrt{f}(T - T_0). \tag{1.9}$$

The first term in the right-hand side of (1.9) is the power of the stress in the total deformation rate and, generally speaking, is not positively determinable.

If along with (1.6) we use the well-known formula (c_σ is the heat capacity at constant stress)

$$c_\sigma = c_\lambda - \frac{\sigma}{\rho_0 \lambda} \frac{\partial \lambda}{\partial T}\Big|_\sigma \tag{1.10}$$

we can obtain an expression for the entropy balance (with the exception of cross sections with strong discontinuities), which can also be represented in the form of the temperature balance:

$$\rho_0 c_\sigma(\lambda)\{\partial(fT)/\partial t + \partial(fvT)/\partial x\} = \sigma f \varphi(\lambda)/6\theta_0(T) - \alpha\sqrt{f}(T - T_0). \tag{1.11}$$

Under these circumstances, from (1.10), taking account of (1.7) and (1.8), we obtain

$$c_\sigma / c_\lambda = 1 + \gamma\lambda\psi'^2/(\lambda\psi')' \quad (\gamma = \mu'/\rho_0 c_\lambda).$$

The first term in the right-hand side of (1.11) represents the dissipation and is positively determined. For the polymer materials under consideration, the value of γ is very small ($\gamma \sim 10^{-2}$).

We note also that formula (1.6) can be written in "divergent" form:

$$\frac{\partial \lambda^{-1}}{\partial t} + \frac{\partial}{\partial x}(v\lambda^{-1}) = \frac{\varphi(\lambda)}{6\theta_0(T)\lambda}. \tag{1.12}$$

The formulas of the present and following sections as $\theta_0 \to \infty$ ($\beta \to \infty$) describe the motion of incompressible ideally elastic rubber in a rod approximation.

2. Weak Discontinuities for the Propagation of Waves in an Elastoviscous Rod

For the closed system of equations (1.1), (1.2), (1.7), and (1.12), taking account of (1.8), the characteristic roots have the form

$$\alpha_{1,2} = v \pm u_S, \ \alpha_3 = v,$$
$$u_S^2 = u_T^2 + \frac{\sigma^2}{\rho_0 c_\lambda T}, \quad u_T^2 = \rho_0^{-1}\lambda^2 \frac{\partial}{\partial \lambda}\left(\frac{\sigma}{\lambda}\right)\Big|_T, \tag{2.1}$$

where u_S and u_T are, respectively, the adiabatic and isothermal velocities of sound in the deformed material. Here $u_S > u_T$ if $\lambda \neq 1$, and $u_S = u_T = u_0(T)$ for $\lambda = 1$, i.e., in the undeformed rod. Here

$$u_0^2(T) = \rho_0^{-1} E(T), \ E(T) = 3n\mu' T. \tag{2.2}$$

As follows from (1.7), $\frac{\partial}{\partial \lambda}\left(\frac{\sigma}{\lambda}\right)\Big|_T = \mu' T \psi''(\lambda) > 0$; this inequality follows from the condition of thermodynamic instability for the potential F and assures the hyperbolic character of the system of equations under consideration.

From (1.7), (2.1), and (2.2) we have

$$\frac{u_T^2}{u_0^2} = \frac{\lambda^2}{3n}\psi''(\lambda), \quad \frac{u_S^2}{u_0^2} = \frac{\lambda^2}{3n}\{\psi'' + \gamma\psi'^2\}, \quad \gamma = \frac{\mu'}{\rho_0 c_\lambda}. \tag{2.3}$$

The characteristic roots α_k for the hyperbolic system coincide with the rates of the propagation of a weak discontinuity $x'_*(t)$. Using, in addition to the system of equations under consideration, the kinetic conditions for compatability, we can obtain an equation interconnecting the dynamic quantities at a weak discontinuity x_* t:

$$f_*(v_* - x'_*)\frac{dv_*}{dt} + \frac{1}{\rho_0 T_*}\frac{d}{dt}(f_*\sigma_* T_*) - \frac{f_*\lambda_*\varphi(\lambda_*)}{6\rho_0\theta_0(T_*)}\frac{\partial\sigma}{\partial\lambda}\Big|_{\lambda_*, T_*} + \frac{\alpha\sqrt{f_*}\sigma_*(T_* - T_0)}{\rho_0^2 c_\lambda T_*} = 0, \tag{2.4}$$

where $f_* = f[x_*(t), t]$; the other quantities are defined analogously. In an isothermal approximation, where $T_* = T_0 =$ const, from (2.4) it follows [1] that

$$f_*(v_* - x'_*)\frac{dv_*}{dt} + \frac{1}{\rho_0}\frac{d(f_*\sigma_*)}{dt} + \frac{f_*\lambda_*\varphi(\lambda_*)d\sigma_*}{6\rho_0\theta_0 \, d\lambda_*} = 0.$$

3. Shock Waves in an Elastoviscous Rod

In the system of equations (1.1), (1.2), (1.7), and (1.12), taking account of (1.8), as has already been noted for the isothermal approximation in [1], strong discontinuities can exist (this system recalls the equations of gasdynamics, where the role of the density is played by the area of the cross section f). Under these circumstances, the structure of the shock wave, along with the zone of inhomogeneity of the stresses in the transverse cross section noted in [1], is also determined by the phenomena of longitudinal thermal conductivity and the presence of a viscous component of the stress in (1.5).

Let $x_0(t)$ be the line of a strong discontinuity in the plane x, t. We select the direction of the x axis so that $x'_0 > 0$. We shall denote by the subscript 1 all quantities ahead of the shock wave ($x = x_0 - 0$) and by the subscript 2 all quantities behind the shock wave ($x = x_0 + 0$). The conditions at the shock waves for the above system of equations have the form

$$[f(v - x_0')] = 0,\quad \rho_0[f(v - x_0')^2] = [\sigma f],$$
$$[\lambda^{-1}(v - x_0')] = 0,\quad \rho_0\left[f(v - x_0')\left(U + \frac{v^2}{2}\right)\right] = [fv\sigma], \tag{3.1}$$

where the standard notation is adopted for the shock wave, $[y] = y_2 - y_1$.

The first equality in (3.1) corresponds to the conservation of the flow of mass with a passage through the shock wave, the second to conservation of the flow of momentum, the third to the conservation of the flow of elastic deformation, and the fourth to the conservation of the energy flux.

We note that, in a fixed system of coordinates, the flow of mass $j < 0$. Since $f_i > 0$, it follows from this that

$$jf_i^{-1} = v_i - x_0' < 0 \qquad (i = 1,\ 2). \tag{3.2}$$

From (3.1) the following relationships can be obtained:

$$f_1\lambda_1 = f_2\lambda_2; \tag{3.3}$$

$$(x_0' - v_i)^2 = \frac{\lambda_i^2}{\rho_0}\frac{[\sigma/\lambda]}{[\lambda]} \qquad (i = 1, 2); \tag{3.4}$$

$$\rho_0[U] = \rho_0 c_\lambda [T] = (1/2)[\lambda](\sigma_1/\lambda_1 + \sigma_2/\lambda_2). \tag{3.5}$$

Relationship (3.3) shows that the passage through the shock wave is accompanied by purely elastic deformation.

From (3.4), using (3.2), we have

$$[v] = -[\lambda]\sqrt{\frac{[\sigma/\lambda]}{\rho_0[\lambda]}}. \tag{3.6}$$

Here, from (3.4) and (3.6) there follows the alternative

$$[\sigma/\lambda] < 0,\quad [\lambda] < 0,\quad [v] > 0; \tag{3.7}$$
$$[\sigma/\lambda] > 0,\quad [\lambda] > 0,\quad [v] < 0. \tag{3.8}$$

The inequalities (3.7), usual for gasdynamics, characterize compression shock waves and are satisfied for the rubberlike materials under consideration, as will be shown below, in the region of compression ($\lambda < 1$) and moderate degrees of elongation. In the region of very large elongations ($\lambda \gg 1$), for rubberlike materials the inequalities (3.8) may obtain, characterizing elongation waves.

From (3.5), taking account of the first equality of (1.7), it follows that

$$\frac{T_2}{T_1} = \frac{2 + \gamma[\lambda]\psi_1'}{2 - \gamma[\lambda]\psi_2'}, \tag{3.9}$$

and we also have the Hugoniot relationship,

$$\frac{\sigma_2}{\lambda_2\psi_2'}(2 - \gamma[\lambda]\psi_2') = \frac{\sigma_1}{\lambda_1\psi_1'}(2 + \gamma[\lambda]\psi_1').$$

Here and in what follows $\psi_i^{(n)} = (d^n\psi/d\lambda^n)_{\lambda=\lambda_i} \qquad (i = 1, 2)$.

Using (1.2) and (3.9), we obtain an expression for the discontinuity of the entropy [S] at the shock wave:

$$\frac{[S]}{c_\lambda} = \ln\left\{\frac{2 + \gamma[\lambda]\psi_1'}{2 - \gamma[\lambda]\psi_2'}\right\} - \gamma[\psi]. \tag{3.10}$$

Formulas (3.9) and (3.10) show that for the dependences $\psi(\lambda)$ given in Fig. 1, the possible region of the existence of shock waves has finite boundaries: $0 < \lambda_- < \lambda < \lambda_+$, where λ_- or λ_+ depend on the parameter λ_1

(or λ_2) ahead of (or behind) the shock wave. As $\lambda \to \lambda_\pm$, where expressions (3.9) and (3.10) become unbounded, the postulation that the heat capacity c_λ is constant becomes untrue, and for the thermodynamic potentials in (1.2) there is a more complex dependence on the temperature.

For shock waves of small intensity, from (3.10) we have

$$\frac{[S]}{c_\lambda} = \frac{\gamma}{12} [\lambda]^3 (\psi_1''' + 3\gamma\psi_1'\psi_1'' + \gamma^2\psi_1'^3) + O([\lambda]^4).$$

As a result of the dissipative character of the shock waves, $[S] > 0$. As follows from Fig. 1, for $\lambda_1 < \lambda_*$, we have $\psi'''_1 < 0$. Then (in view of the smallness of γ), waves of small intensity can be shock waves for $\lambda_1 < \lambda_*$ only if $[\lambda] < 0$ (compression waves). For $\lambda_1 > \lambda_*$ we have $\psi'''_1 > 0$, and waves of small intensity can be shock waves only for $[\lambda] > 0$.

The results of an analysis of the behavior of [S] with arbitrary values of $[\lambda]$ as a function of λ_1 and λ_2 are shown qualitatively in Fig. 2 for the real potential $\psi(\lambda)$ shown in Fig. 1 by the continuous lines. We consider two cases separately.

In the first case, where loading waves are being considered, it is convenient to study the dependence of [S] on λ_2 for fixed values of λ_1 (Fig. 2a-c).

In the second case, where unloading waves are being studied, it is convenient to study the dependence of [S] on λ_1 for fixed values of λ_2 (Fig. 2a'-c').

Here, for the plots of Fig. 2, use was made of the antisymmetry of the function $[S] = g(\lambda_1, \lambda_2)$, i.e., $g(\lambda_2, \lambda_1) = -g(\lambda_1, \lambda_2)$, which follows directly from (3.10).

In addition to the physical condition for the existence of shock waves, $[S] > 0$, in the situation under consideration, where $\psi'''(\lambda)$ changes sign, the question of their stability must still be examined [7, 8].

The shock wave will be stable if the following inequalities are simultaneously satisfied:

$$u_{S_1} < x_0' < u_{S_2}, \tag{3.11}$$

since only in this case will small perturbations behind the shock wave overtake it and supply energy to it, while small perturbations ahead of the shock wave, moving more slowly than it, cannot take energy away from the shock wave.

From formulas (2.1)-(2.3) and (3.4), the following relationships can be obtained:

$$\begin{aligned}(x_0' - v_1)^2 - (u_{S_1} - v_1)^2 &= -\lambda_1^2 u_0^2 (T_1) \frac{2 + \gamma [\lambda] \psi_1'}{3n\gamma c_\lambda [\lambda]} \frac{\partial}{\partial \lambda_1}[S],\\ (u_{S_2} - v_2)^2 - (x_0' - v_2)^2 &= \lambda_2^2 u_0^2 (T_1) \frac{2 + \gamma [\lambda] \psi_1'}{3n\gamma c_\lambda [\lambda]} \frac{\partial}{\partial \lambda_2}[S].\end{aligned} \tag{3.12}$$

From (3.12) it follows that the conditions for stability (3.11) will be satisfied if the following relationships hold:

$$\operatorname{sgn}\left\{\frac{\partial}{\partial \lambda_2}[S]\right\} = \operatorname{sgn}[\lambda],\ \operatorname{sgn}\left\{\frac{\partial}{\partial \lambda_1}[S]\right\} = -\operatorname{sgn}[\lambda]. \tag{3.13}$$

Let us now examine the question of the existence and stability of loading and unloading shock waves, propagating along rods made of a rubberlike material.

The dependence of [S] on λ_2 for $\lambda_1 < \lambda_*$ and for the real potential $\psi(\lambda)$ is illustrated qualitatively in Fig. 2a and the dependence of [S] on λ_1 for $\lambda_2 < \lambda_*$ in Fig. 2a'. Here λ_* is a point of inflection on the dependence $\psi'(\lambda)$ (see Fig. 1).

As can be seen from Fig. 2a, loading shock waves exist in this case ($[S] > 0$) for $\lambda_2 < \lambda_1$ [compression shock waves, for which the inequalities (3.7) are satisfied], and exist also for $\lambda_2 > \lambda_1^0 > \lambda_1$ [elongation shock waves, for which the inequalities (3.8) are satisfied]. As follows from (3.13), both of these types of shock waves are stable. In the interval $\lambda_1 \leq \lambda_2 \leq \lambda_1^0$, there exist only weak isentropic loading waves.

From Fig. 2a' it can be seen that unloading waves exist only in the region $\lambda_2 < \lambda_1 < \lambda_2^0$. However, for $\lambda_2 < \lambda_1 < \lambda_m$ [where λ_m is the point of the maximum $[S](\lambda_2)|_{\lambda_2 = \text{const}}$], as follows from (3.13), they are stable, and, for $\lambda_m < \lambda_1 < \lambda_2^0$, they are unstable. Thus, unloading shock waves actually exist only in the interval

$\lambda_2 < \lambda_1 \leq \lambda_m$. Outside this interval, there exist only weak isentropic unloading waves. Stable shock waves and weak loading and unloading waves are shown in Fig. 2 by the heavy lines.

Only stable loading shock waves exist for $\lambda_1 = \lambda_1^0 \approx \lambda_*$ (see Fig. 2b). Here, if $\lambda_2 < \lambda_1$, these are compression waves with satisfaction of the inequalities (3.7); if $\lambda_2 > \lambda_1$, they are elongation shock waves with the satisfaction of inequalities (3.8). Only weak isentropic unloading waves exist for $\lambda_2 = \lambda_2^0 \approx \lambda_*$ (see Fig. 2b').

The dependence of [S] on λ_2 for a fixed value of $\lambda_1 > \lambda_*$ is shown in Fig. 2c. Stable loading shock waves exist for $\lambda_2 < \lambda_1^0$ [compression shock waves with the satisfaction of inequalities (3.7)] and for $\lambda_2 > \lambda_1$ [elongation shock waves with satisfaction of the inequalities (3.8)]. In the interval $\lambda_1^0 < \lambda_2 < \lambda_1$ there exist only weak isentropic loading waves.

Shock unloading waves for $\lambda_2 > \lambda_*$ exist, as is shown in Fig. 2c', in the interval $\lambda_2^0 < \lambda_1 < \lambda_2$. However, they are stable, as follows from (3.13), only in the interval $\lambda_m < \lambda_1 < \lambda_2$, where λ_m is a maximum on the dependence $[S]\ (\lambda_1)|_{\lambda_2 = \text{const}}$. Outside this interval there exist only weak isentropic unloading waves.

Dissipative phenomena (relaxation, heat transfer) can lead to a case where the situations illustrated in Fig. 2 will vary with the time; in particular, one situation may replace another. In a heat-insulated rod, to which energy is applied only for the course of a finite interval of time which is very small in comparison with the time of the existence of the wave, these changes are directed toward the side of a decrease in the intensity of the waves.

4. Examples

We first consider loading waves, propagating over a homogeneous unloaded rod having a temperature $T_1 = T_0$.

In the given case, $\lambda_1 = 1$, $f_1 = \text{const}$, $\sigma_1 = 0$, $T_1 = T_0$, and, without limiting the generality, we can assume that $v_1 = 0$. This situation is obviously illustrated in Fig. 2a.

From Fig. 2a it follows that, for $\lambda_2 < 1$ ($\lambda_1 = 1$), there exist compression shock waves. Under these circumstances, from (3.3)-(3.5) we have

$$f_2 = f_1 \lambda_2^{-1},\ T_2/T_0 = \{1 - (\gamma/2)(\lambda_2 - 1)\psi_2'\}^{-1}$$
$$x_0' = u_0(T_0)(\psi_2')^{1/2}\{3n\lambda_2(\lambda_2 - 1)(1 + (\gamma/2)(1 - \lambda_2)\psi_2')\}^{-1/2},\ v_2 = x_0'(1 - \lambda_2). \quad (4.1)$$

Since $\lambda_2 < 1$, then $\psi'_2 < 0$, and from (4.1) we have

$$f_2 > f_1,\ T_2 > T_0,\ x_0' > v_2 > u_0(T_0) > 0.$$

For sufficiently smooth initial data, the front of the compression wave in the given case will be twisted, exhibiting a tendency toward the appearance of a strong discontinuity.

In the region $1 < \lambda_2 < \lambda_1^0$, as follows from Fig. 2a, there exist only weak discontinuities; therefore, if the degree of the initial elongation $\lambda_2(0) < \lambda_1^0$, then, even with discontinuous conditions, the front of the wave will be washed out.

In the region $\lambda_2 > \lambda_1^0$, i.e., with sufficiently large original elongation deformations, as follows from Fig. 2a, there arise elongation shock waves, irregardless of the smoothness of the original distribution of the sought values. In this case, all the values behind the shock wave will again be described by formulas (4.1) for $\lambda_2 \gg 1$. From this it follows that $f_2 < f_1$, $T_2 > T_0$, $x'_0 > u_0(T_0)$, and $v_2 < 0$.

Dissipative phenomena (relaxation stresses, heat transfer) behind the front of a shock wave introduce considerable changes into elongation and compression shock waves with loading. In both cases, relaxation of the stresses leads to a change in the parameter λ_2 (with compression, λ_2 increases and, with elongation, decreases). In the case of a compression shock wave, this fall in the intensity continues right up to $\lambda_2 \to 1$; in this case, the compression shock wave is retained. In the case of an elongation shock wave, such a fall in the intensity with the retention of a strong discontinuity exists only up to the moment of time t_*, when $\lambda_2(t_*)$ becomes equal to λ_1^0. For $t > t_*$, $\lambda_2 < \lambda_1^0$ and the shock wave will be washed out due to instability. Thus, in distinction from a compression shock wave, an elongation shock wave exists only for a finite time t_*, then becoming unstable.

Taking account of lateral heat transfer, lowering the temperature T_2 behind the shock wave brings about an increase (in comparison with adiabatic deformation) in the relaxation time and, as a result of this, somewhat slows down the course of the relaxation processes. Nevertheless, the qualitative picture for both types of loading shock waves under consideration remains valid.

As follows from (4.1), all the values at the shock wave are completely determined by the dependence $\lambda_2(t)$. This dependence is found, as is well known, from the solution of the complete nonlinear boundary-value problem in the region behind the shock wave. If, behind the front of the shock wave, dissipative phenomena can be neglected, this problem is made easier. For example, formulas (4.1) yield an exact solution of the problem of the pulsed compression of a semibounded elastic rod (λ_2 = const).

Let us now consider unloading waves, propagating over a homogeneous loaded rod relaxing under isothermal conditions.

In a purely elastic case ($\theta_0 \to \infty$ or $\beta \to \infty$), $\lambda_2 = 1$, while λ_1, f_1, and T_1 are known quantities, $v_1 = 0$, the situation is illustrated in Fig. 2a', where, in the interval $1 < \lambda_1 < \lambda_m$, there exist unloading shock waves of an elongated sample. The relationships at the shock wave have the form

$$f_2 = \lambda_1 f_1,\ T_2 = T_1\{1 - \gamma(\lambda_1 - 1)\psi_1'\},$$
$$x_0' = u_0(T_1)\lambda_1\sqrt{\frac{\psi_1'}{3n(\lambda_1 - 1)}},\quad v_2 = x_0'\frac{\lambda_1 - 1}{\lambda_1}. \tag{4.2}$$

We note that, according to (4.2), we have $T_2 < T_1$. It can be shown that this drop in the temperature is less than for unloading under isentropic conditions.

Relationships (4.2) constitute an exact solution of the problem for the adiabatic unloading of an elastic rod, i.e., they describe the distribution of the sought values also behind the shock wave.

Taking account of relaxation phenomena can qualitatively change the picture of all the distributions behind a shock wave. In the present case, as before, f_1 = const, T_1 = const, and $v_1 = 0$; however, the value of $\lambda_1(t)$ is determined from the relaxation equation

$$\frac{d\lambda_1}{dt} + \frac{\lambda_1\varphi(\lambda_1)}{6\theta_0(T_1)} = 0,\ \lambda_1(0) = \lambda_1^0. \tag{4.3}$$

This circumstance leads to a situation in which, behind the shock front, some value of the deformation is formed, $\lambda_2(t) < \lambda_1(t)$, which is determined from a solution of the complete nonlinear dynamic problem as a whole. For $\lambda_1 < \lambda_m$, the relationships at the shock wave are described by the overall formulas (3.3)-(3.6), taking account of $v_1 = 0$.

In the case where $x'_0(\lambda_1^0)\theta_0 L^{-1} \gg 1$ (L is the length of the rod in the deformed state), i.e., the characteristic time of the propagation of a loading shock wave over the rod is less than the relaxation time, we can write out the principal terms of the asymptotic solution behind the shock wave:

$$\lambda(x, t) \approx \lambda_2 = 1,\ \sigma(x, t) \approx \sigma_2 = 0,\ v(x, t) \approx v_2(t), \tag{4.4}$$

in which the values of f_2, T_2, v_2, and x'_0, satisfy (4.2) with $\lambda_1(t)$ determined from (4.3); for f(x, t) we will obtain the Cauchy problem

$$\frac{\partial f}{\partial t} + v(t)\frac{\partial f}{\partial x} = 0,\ f\big|_{x=x_0(t)} = f_2(t) = f_1\lambda_1(t), \tag{4.5}$$

having the solution

$$f = f_1\lambda_1\{l^{-1}(x - a(t))\},\ a(t) = \int_0^t v_2(\tau)d\tau,\ l(t) = x_0(t) - a(t). \tag{4.6}$$

When $x'_0(\lambda_1^0)\theta_0 L^{-1} \lessgtr 1$, formulas (4.4)-(4.6) are unsuitable and we must turn to a numerical solution of the problem as a whole. The most important fact in the present case is that a zone of compression may arise in the relaxing rod behind an unloading shock wave. In actuality, if it is postulated that, behind the shock wave,

$$df/dt \equiv \partial f/\partial t + v\partial f/\partial x > 0,$$

then from the first equation of (1.1) it follows that, behind the shock wave, $\partial v/\partial x < 0$. The latter can lead to a situation in which $\lambda_2 < 1$, $\sigma_2 < 0$, i.e., in some region behind the shock wave there may be relaxation of an inhomogeneous compressed material.

We note that the formulas given above, describing unloading shock waves, are valid only if $\lambda_1^0 < \lambda_m$ (see Fig. 2a'). In the contrary case, where $\lambda_1^0 > \lambda_m$, at the start of the unloading process, shock waves will not exist, and only in a certain time after the start of the unloading process, in a medium with a sufficiently small relaxation time (or for a very long rod), will the weak wave arising at the start go over into a shock wave, whose intensity will than fall further with time.

LITERATURE CITED

1. Z. N. Kuzina, A. I. Leonov, and A. A. Markov, "Nonlinear waves in an elastoviscous rod and the problem of the impact of a rod of finite length on a rigid barrier," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1978).
2. P. J. Flory, Principles of Polymer Chemistry, Cornell University Press, Ithaca (1953).
3. Boa-Teh Chu, "Transverse shock waves in incompressible elastic solids," J. Mech. Phys. Solids, 15, 1-14 (1967).
4. P. J. Blatz, S. C. Sharda, and N. W. Tshoegl, "Strain energy function for rubberlike materials, based on a generalized measure of strain," Trans. Soc. Rheol., 18, No. 1, 145-161 (1974).
5. A. I. Leonov, "Nonequilibrium thermodynamics and rheology of elastoviscous polymer media," Preprint of Lecture at the International School on the Problem of Heat and Mass Transfer, A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk, April 16-28 (1975).
6. A. I. Leonov (A. J. Leonov), "Nonequilibrium thermodynamics and rheology of viscoelastic polymer media," Rheologica Acta, 15, No. 2, 85-98 (1976).
7. L. D. Landau and E. M. Lifshits, The Mechanics of Continuous Media [in Russian], Gostekhteorizdat, Moscow (1954).
8. G. Ya. Galin, "Shock waves in media with an arbitrary equation of state," Dokl. Akad. Nauk SSSR, 119, No. 6, 1106-1109 (1958).

DISPERSION OF THE VELOCITY AND SCATTERING OF ULTRASONIC WAVES IN COMPOSITE MATERIALS

A. A. Usov and T. D. Shermergor

UDC 534.514

The scattering of waves at the inhomogeneities of a medium can be calculated by various methods. An analysis of the most frequently used approximations was made in [1, 2]. The scattering coefficient of an ultrasonic wave in compositite materials was calculated in [3-6]. In [3], the smallness of one of the components was assumed, while, in [6], only the asymptote of long and short waves were calculated. An attempt at the calculation of the scattering coefficient of longitudinal and transverse ultrasonic waves over the whole range of wavelengths was made in [4, 5]. The calculation was made under the approximation of taking account of pairwise correlations between the moduli of elasticity and the density. In [4], the calculations were made using a Gaussian distribution for the coordinate parts of the binary correlation functions, which does not relate to composite materials, and, in [5], the explicit form of a function enabling a transition from asymptote of long waves to a short-wave asymptote is not given. In addition, neither of the above-cited pieces of work took into consideration the distribution of the velocity of the propagating wave.

A calculation of the scattering coefficient and the dispersion of the velocity of longitudinal waves over the whole range of wavelengths, with arbitrary concentrations of the components, is given below.

§ 1. We renormalize the equations of motion using a method developed in [7-9]:

$$L_{il}u_l = 0,\ L_{il} \equiv \nabla_k\lambda_{iklm}\nabla_m + \rho\omega^2\delta_{il},$$

where u is the vector of the displacement; λ_{iklm} is the tensor of the moduli of elasticity; ρ is the density of the medium; ω is the cyclic frequency.

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 145-152, May-June, 1978. Original article submitted May 3, 1977.

 0021-8944/78/1903-0398$07.50